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# The role of the Beltrami parametrization of complex structures in 2-d Free Conformal Field Theory

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ABSTRACT. This talk gives a review on how complex geometry and a Lagrangian formulation of 2-d conformal field theory are deeply related. In particular, how the use of the Beltrami parametrization of complex structures on a compact Riemann surface fits perfectly with the celebrated locality principle of field theory, the latter requiring the use infinite dimensional spaces. It also allows a direct application of the local index theorem for families of elliptic operators due to J.-M. Bismut, H. Gillet and C. Soulé. The link between determinant line bundles equipped with the Quillen's metric and the so-called holomorphic factorization property will be addressed in the case of free spin  $j$  b-c systems or more generally of free fields with values sections of a holomorphic vector bundles over a compact Riemann surface.

## 1. Introduction

In the last decade now, both string field and bidimensional conformal field theories have enjoyed a large popularity in physics as well as in mathematics, for a mathematical reading see e.g. [1] and references therein. On the mathematical side, it seems rather remarkable how some problems of these special classes of low dimensional field theories helped to shed some new light on the classical geometry of Riemann surfaces. Meanwhile, a refined version of the Atiyah-Singer index theorem for families due to Bismut, Freed, Gillet and Soulé has been completed [2, 3, 4, 5].

However, on the physical side, it is fair to note that quantization schemes over moduli space have been widely preferred to the conventional schemes (Feynman) grounded on some locality principle. In particular, one of the main features of the bidimensional conformal field theories is the so-called holomorphic-antiholomorphic factorization property. Moreover, derivations of the factorization property from conventional field theory scheme are so far limited to infinite dimensional spaces.

In this talk, we shall report on how the holomorphic factorization property can be derived for free conformal fields on a compact Riemann surface without boundary and of genus greater than one. It comes from resummed renormalized perturbation theory for vacuum functionals considered as defined on the infinite

dimensional space of complex structures on the Riemann surface (the well known Beltrami parametrization) or on some holomorphic vector bundle over it [6, 7, 8]. This provides a local version of the Belavin-Knizhnik theorem [9] which is achieved by using the above mentioned local index theorem for families by Bismut et al. This gives an important bridge with the locality principle of the Euclidean version of field theory and the resummed renormalized perturbation theory.

In Section 2, the free bosonic string will be recalled in the metric set up. Section 3 describes the Beltrami parametrization of complex structures and some of its advantages. Section 4 will concern the application of the local index theorem in the version given By Bismut et al. Finally some concluding remarks are gathered in section 5.

## 2. The classical String action

Throughout this talk we will be concerned with the Euclidean framework. In order to fix the idea one starts with the standard classical string action, [10, 11, 12], see also Bost [1],

$$(2.1) \quad S(X, g) = \frac{1}{8} \int_{\Sigma} d^2x \left( \sqrt{g} X \Delta(g) X \right)(x) = -\frac{1}{8} \int_{\Sigma} d^2x \left( X \partial_{\alpha} \hat{g}^{\alpha\beta} \partial_{\beta} X \right)(x) ,$$

where  $x = (x^1, x^2)$  denotes a set of local coordinates on the compact Riemann surface  $\Sigma$  without boundary,  $g$  is a Riemannian metric on this surface with the associated Laplace-Beltrami operator, see e.g. [13], acting on scalar functions,

$$(2.2) \quad \Delta(g) = \frac{-1}{\sqrt{g}} \partial_{\alpha} \hat{g}^{\alpha\beta} \partial_{\beta} , \quad \partial_{\alpha} = \frac{\partial}{\partial x^{\alpha}} ,$$

acting on string coordinates  $X(x) \in \mathbf{R}^D$ . The second integral in (2.1) displays the Weyl invariant unimodular non-degenerate quadratic form  $\hat{g}$  (density of metrics) related to the conformal class of the metric  $g$  [14, 15],

$$(2.3) \quad \hat{g}_{\alpha\beta} \equiv \frac{1}{\sqrt{g}} g_{\alpha\beta} , \quad \hat{g} \equiv \det \hat{g} = 1 ,$$

showing the obvious invariance of (2.1) under Weyl transformations. See [15, 16, 17, 18] for some historical background.

## 3. From metrics to complex structures

In this section the relationship in two dimensions between conformal classes of metrics and Beltrami parametrization of complex structure is recalled.

**3.1. The smooth change of complex coordinates.** We shall right away introduce a complex analytic atlas with local coordinates  $(z, \bar{z})$ <sup>1</sup> corresponding to the reference complex structure. We then parametrize locally the metric  $g$  according to [19],

$$(3.1) \quad ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} = \rho_{z\bar{z}} |dz + \mu_{\bar{z}}^z d\bar{z}|^2 ,$$

where  $\mu_{\bar{z}}^z \equiv \mu$  is the local representative of the Beltrami differential ( $|\mu| < 1$ ) seen as a  $(-1, 1)$  conformal field, which parametrizes the conformal class of the metric

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<sup>1</sup>We shall omit the index denoting the open set where these coordinates are locally defined since all formulae will glue through holomorphic changes of coordinates.

$g$  and  $\rho_{z\bar{z}} \equiv \rho > 0$  is the coefficient of a positive real valued type (1,1) conformal field.

Our compact Riemann surface  $\Sigma$  without boundary being now endowed with the analytic atlas with local coordinates  $(z, \bar{z})$ , let  $(Z, \bar{Z})$  be the local coordinates of another holomorphic atlas corresponding to the Beltrami differential  $\mu$ ,

$$(3.2) \quad dZ = \lambda(dz + \mu d\bar{z}) \quad \implies \quad \partial_{\bar{Z}} = \frac{\bar{\partial} - \mu \partial}{\lambda(1 - \mu\bar{\mu})}$$

where  $\lambda = \partial_z Z \stackrel{\text{Def}}{=} \partial Z$  is an integrating factor fulfilling <sup>2</sup>

$$(3.3) \quad (d^2 = 0) \iff (\bar{\partial} - \mu \partial) \ln \lambda = \partial \mu .$$

Solving the above Pfaff system (3.2) is equivalent to solving locally the so-called Beltrami equation

$$(3.4) \quad (\bar{\partial} - \mu \partial) Z = 0 .$$

According to Bers, see e.g. [19], the Beltrami equation (3.4) always admits as a solution a quasiconformal mapping with dilatation coefficient  $\mu$ . One thus remarks that  $Z$  is a (non-local) holomorphic functional of  $\mu$  as well as the integrating factor  $\lambda$ . However, the solution of the Beltrami equation define a smooth change of local complex coordinates  $(z, \bar{z}) \longrightarrow (Z, \bar{Z})$  which preserves the orientation (the latter condition secures the requirement  $|\mu| < 1$ ), so that  $(Z, \bar{Z})$  defines a new system of complex coordinates with  $Z \longrightarrow z$  when  $|\mu| \longrightarrow 0$ .

**3.2. The classical String action revisited.** In terms of the  $(Z, \bar{Z})$  complex coordinates which by virtue of (3.2) turns out to be isothermal coordinates for the metric  $g$  by defining the non-local metric

$$(3.5) \quad \rho_{Z\bar{Z}} \equiv \frac{\rho}{\lambda \bar{\lambda}} .$$

In particular, the quadratic form (3.1), the volume form, the scalar curvature and the scalar Laplacian, respectively write,

$$(3.6) \quad \begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta = \rho_{Z\bar{Z}} |dZ|^2, & \sqrt{g} &= \rho(1 - \mu\bar{\mu}), \\ d^2x \sqrt{g} &= \frac{d\bar{Z} \wedge dZ}{2i} \rho_{Z\bar{Z}}, & \Delta(g) &= \frac{-4}{\rho_{Z\bar{Z}}} \partial_Z \partial_{\bar{Z}}, \end{aligned}$$

in such way that the classical string action (2.1) now reads

$$(3.7) \quad S(X, \mu, \bar{\mu}) = -\frac{1}{2} \int_{\Sigma} \frac{d\bar{Z} \wedge dZ}{2i} (X \partial_Z \partial_{\bar{Z}} X)(Z, \bar{Z}) ,$$

which is nothing but the usual string action in the conformal gauge [12] where the dependence in  $\mu$  is hidden within the complex coordinate  $Z$  in an *a priori* non-local way. The latter is explicitly restored by going back to the  $(z, \bar{z})$  complex coordinates [16, 18] thanks to both (3.2) and (3.3),

$$(3.8) \quad S(X, \mu, \bar{\mu}) = -\frac{1}{2} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \frac{1}{2} \left( X (\bar{\partial} - \partial \mu) \frac{1}{1 - \mu\bar{\mu}} (\partial - \bar{\mu} \bar{\partial}) X \right) (z, \bar{z}),$$

---

<sup>2</sup>From now on, we shall reserve  $\partial$ 's for  $\partial \equiv \partial_z$ ,  $\bar{\partial} \equiv \partial_{\bar{z}}$ .

which turns out to be local in the complex structure  $\mu$  thanks to the disappearance of the non-local integrating factor upon using eq.(3.3).  $\mu$  and  $\bar{\mu}$  are sources for the two components  $\Theta$  (respectively  $\bar{\Theta}$ )

$$(3.9) \quad \Theta(z, \bar{z}) \equiv \frac{\delta S}{\delta \mu(z, \bar{z})} \Big|_{\mu=\bar{\mu}=0} = -\frac{1}{2}(\partial X)^2(z, \bar{z}) ,$$

of the energy-momentum tensor.

Roughly speaking the quantization à la Feynman of the string model yields the formal computation of the logarithm of the determinant of the scalar Laplacian by a formal power series in  $\mu, \bar{\mu}$ ,  $\Gamma(\mu, \bar{\mu})$ , with coefficients distributions, the correlation functions of the above two components of the energy-momentum tensor. The question of computing

$$\Gamma(\mu, \bar{\mu}) = \text{“} \ln \det \partial_Z \partial_{\bar{Z}} \text{”}$$

is addressed in the next section.

#### 4. The application of the local index theorem for families

**4.1. The geometrical data.** The *full* geometrical construction goes as follows. We first choose a *prescribed* compact Riemann surface  $(\Sigma, (z, \bar{z}))$  without boundary where  $\Sigma$  is a smooth compact surface without boundary and  $(z, \bar{z})$  denotes a *fixed* local set of complex analytic coordinates on  $\Sigma$ . Now we are in a position to define Beltrami differentials over  $\Sigma$ . That is, if  $\kappa$  denotes the canonical holomorphic line bundle over  $\Sigma$ , we consider the bundle of smooth  $(-1, 1)$ -differentials over  $\Sigma$

$$(4.1) \quad \kappa^{-1} \otimes \bar{\kappa} \longrightarrow \Sigma,$$

and sections of this bundle are given by

$$(4.2) \quad \hat{\mu} = \mu_{\bar{z}}^z d\bar{z} \otimes \partial_z$$

where  $\mu_{\bar{z}}^z \equiv \mu$  is the coefficient of the  $(-1, 1)$ -differential patching under holomorphic change of charts  $(U, z) \rightarrow (V, w)$ ,  $w = w(z)$  as

$$\mu_{\bar{w}}^w = \frac{w'}{\bar{w}'} \mu_{\bar{z}}^z .$$

In order to obtain the space of Beltrami differentials parametrizing complex structures on the surface  $\Sigma$  we must impose some restrictive conditions on the bundle, i.e. on the coefficient  $\mu$ . We have  $|\mu_{\bar{z}}^z| < 1$  and this absolute value is independent of coordinates, consequently the  $L_\infty$ -norm of  $\mu$  is well defined [20].

Let  $B$  be the infinite dimensional space of Beltrami differentials. This space has nice topological properties [20]: it is a contractible complex manifold, convex and circled but non compact. Then any bundle over  $B$  is topologically trivial.

**4.2. The holomorphic family of compact Riemann surfaces.** Let  $\pi : B \times \Sigma \longrightarrow B$  be the proper holomorphic map of compact Riemann surfaces. For every  $\mu \in B$ , let  $\Sigma_\mu = \pi^{-1}\{\mu\}$  be the fiber over  $\mu$ , that is a Riemann surface made by  $\Sigma$  endowed with the complex structure given by  $\mu$ . The local coordinates of the corresponding atlas will be denoted  $(Z_\mu, \bar{Z}_\mu)$  with

$$(4.3) \quad dZ_\mu = \lambda_\mu(dz + \mu d\bar{z}) ,$$

where  $\lambda_\mu = \partial Z_\mu$  is an integrating factor which fulfills

$$(4.4) \quad (\bar{\partial} - \mu \partial) \ln \lambda_\mu = \partial \mu .$$

REMARK 4.1. Note that the fiber  $\Sigma_\mu$  is equivalent to the pair  $(\Sigma, \mu)$ . More precisely, given a Beltrami differential  $\mu$  on  $\Sigma$  then surface  $\Sigma$  is endowed with the new complex structure corresponding to that  $\mu$ . So the new Riemann surface obtained is the fiber  $\Sigma_\mu$ . The *reference* Riemann surface  $(\Sigma, (z, \bar{z}))$  is the 0-fiber  $\Sigma_0 = (\Sigma, 0)$ .

The local coordinates on  $B \times \Sigma$  as smooth ( $C^\infty$ ) trivial bundle are  $(\mu, \bar{\mu}, z, \bar{z})$  if all fibers are seen to be modeled on  $\Sigma$ . The total differential is then

$$(4.5) \quad D = d + \partial_B + \bar{\partial}_B$$

with  $d = dz\partial_z + d\bar{z}\partial_{\bar{z}}$  and  $\partial_B, \bar{\partial}_B$  are the differentials<sup>3</sup> according to the complex structure of  $B$ . We want to exhibit the complex analytic structure on  $B \times \Sigma$  induced by both those of  $B$  and  $\Sigma$  compatible with the  $C^\infty$ -structure. Indeed in the  $C^\infty$ -bundle structure it is defined by the local coordinates  $(\mu, \bar{\mu}, Z, \bar{Z})$  where the  $Z$  coordinates are the generic coordinates on the fiber  $\Sigma$ . In other words, we obtain the local complex analytic coordinates on the fiber  $\Sigma_\mu$  above  $\mu$  by solving (locally) the Beltrami equation (see eq.(4.3))

$$(4.6) \quad (\bar{\partial} - \mu\partial)Z = 0,$$

and picking out solutions  $Z_\mu$  holomorphic in  $\mu$ .

Let  $T(B \times \Sigma)$  be the tangent bundle over  $B \times \Sigma$ . We restrict ourselves to the vertical holomorphic tangent bundle  $\kappa_\Sigma^{-1}$  which is a line bundle over  $B \times \Sigma$  and its restriction to a fiber is isomorphic to the holomorphic tangent bundle of this fiber. Notice that  $\kappa_\Sigma$  is the canonical holomorphic line bundle of *vertical*  $(1, 0)$ -forms on  $B \times \Sigma$ . Later on we shall be concerned with the *compact vertical cohomology and the integration along the fiber*  $\Sigma$  [21].

Now for every  $\mu \in B$ , due to the dimension of the fiber we can introduce a Kähler metric [6]

$$(4.7) \quad \rho_{Z_\mu \bar{Z}_\mu} = \frac{\rho_{z\bar{z}}}{\lambda_\mu \bar{\lambda}_\mu},$$

on  $\Sigma_\mu$  depending smoothly on  $\mu$  and where  $\rho_{z\bar{z}} \equiv \rho$  is the Kähler metric on the 0-fiber.

Then the bundle  $\kappa_\Sigma^{-1}$  is endowed with the Hermitian metric

$$(4.8) \quad \rho_{Z\bar{Z}} = \frac{\rho_{z\bar{z}}}{\lambda\bar{\lambda}}.$$

Moreover the induced norm on the Hermitian bundle  $\kappa_{\Sigma_\mu}^{\otimes j} \otimes \bar{\kappa}_{\Sigma_\mu}^{\otimes \bar{j}}$  of  $(j, \bar{j})$ -differentials  $a(dZ_\mu)^j (d\bar{Z}_\mu)^{\bar{j}}$  on  $\Sigma_\mu$  is

$$(4.9) \quad \begin{aligned} \|a\|^2 &\stackrel{\text{Def}}{=} \|\alpha\|^2 = \int_\Sigma \frac{d\bar{Z}_\mu \wedge Z_\mu}{2i} |a|^2 \rho_{Z_\mu \bar{Z}_\mu}^{1-j-\bar{j}} \\ &= \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} (1 - \mu\bar{\mu}) |\alpha|^2 \rho^{1-j-\bar{j}} \end{aligned}$$

---

<sup>3</sup>The reader must keep in mind that  $B$  is an infinite dimensional manifold which will be treated formally the time being.

where thanks to the change of coordinates  $(z, \bar{z}) \mapsto (Z_\mu, \bar{Z}_\mu)$  on  $\Sigma$

$$(4.10) \quad a = \frac{\alpha}{\lambda_\mu^j \bar{\lambda}_\mu^j}.$$

This last relation (4.10) allows to consider the Hermitian bundle of  $(j, \bar{j})$ -differentials over the 0-fiber.

**4.3. The determinant line bundle.** Our present task is to compute the first Chern class of the  $\bar{\partial}$ -determinant line bundle over  $B$  by using the local form of the Atiyah-Singer index theorem for families of elliptic operators given by [1, 2, 3, 4].

We are interested in the following *holomorphic* family of  $\bar{\partial}$ -operators over the compact Riemann surfaces  $\Sigma$  fibers of  $\pi$  : let  $E = \kappa_\Sigma^{\otimes j}$  be the bundle whose restriction to the fiber  $\Sigma_\mu$  is the bundle of smooth  $(j, 0)$ -differentials on this fiber and if  $\Gamma(\Sigma_\mu, E)$  denotes the smooth sections of the bundle  $E$  over  $B \times \Sigma$ , we consider

$$(4.11) \quad \bar{\partial}_\mu : \Gamma(\Sigma_\mu, E) \longrightarrow \Gamma(\Sigma_\mu, E \otimes \bar{\kappa}_\Sigma),$$

the operator  $\bar{\partial}_\mu$  mapping  $(j, 0)$ -differentials to  $(j, 1)$ -differentials on  $\Sigma_\mu$ .

In particular if  $c(dZ_\mu)^j$  is such a  $(j, 0)$ -differential then by the construction (4.9) together with (4.10) we get

$$(4.12) \quad \begin{aligned} \|\bar{\partial}_\mu c\|^2 &= \int_\Sigma \frac{d\bar{Z}_\mu \wedge Z_\mu}{2i} |\partial_{\bar{Z}_\mu} c|^2 \rho_{Z_\mu \bar{Z}_\mu}^{-j} \\ &= \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} \frac{1}{(1 - \mu\bar{\mu})} |(\bar{\partial} - \mu\partial - j(\partial\mu)) \gamma|^2 \rho^{-j}, \end{aligned}$$

after use of

$$(4.13) \quad \partial_{\bar{Z}_\mu} = \frac{\bar{\partial} - \mu\partial}{\bar{\lambda}_\mu(1 - \mu\bar{\mu})},$$

together with eq.(4.4), and where we have set by a local rescaling,

$$(4.14) \quad c = \frac{\gamma}{\lambda_\mu^j},$$

in order to define the corresponding  $(j, 0)$ -differential with respect to the local complex coordinates  $(z, \bar{z})$ .

**PROPOSITION 4.2 ([6]).** *According to the formula (4.12) the family of (positive) Laplacians  $\Delta_j$  on the compact Riemann surfaces  $\Sigma$  is defined through*

$$(4.15) \quad \|\bar{\partial}_\mu c\|^2 = \langle \gamma | \Delta_{j,\mu} \gamma \rangle,$$

where  $\Delta_{j,\mu} = \mathcal{T}_{j,\mu}^* \mathcal{T}_{j,\mu}$  with,

$$(4.16) \quad \begin{aligned} \mathcal{T}_{j,\mu} &= \frac{1}{1 - \mu\bar{\mu}} (\bar{\partial} - \mu\partial - j(\partial\mu)) \\ \mathcal{T}_{j,\mu}^* &= \frac{-1}{\rho^{1-j}(1 - \mu\bar{\mu})} \overline{(\bar{\partial} - \mu\partial - (1-j)(\partial\mu))} \frac{1}{\rho^j} = -\rho^{j-1} \bar{\mathcal{T}}_{1-j,\mu} \rho^{-j} \end{aligned}$$

where the adjoint is with respect to the norm of  $(j, 1)$ -differentials, cf eq.(4.12).

Note that for  $j = 0$ , the scalar Laplacian of the string action (3.8) is recovered.

PROPOSITION 4.3 ([6]). *Through the smooth change of local complex coordinates  $(Z, \bar{Z}) \mapsto (z, \bar{z})$  the holomorphic family  $\bar{\partial}_\mu$  (4.11) of  $\bar{\partial}$ -operators on compact Riemann surfaces  $\Sigma$  is equivalent to the (only) smooth family  $\mathcal{T}_j$  of  $\bar{\partial}$ -operators on the 0-fiber  $(\Sigma, (z, \bar{z}))$ ,*

$$(4.17) \quad \mathcal{T}_{j,\mu} : \Gamma(\Sigma, \kappa^{\otimes j}) \longrightarrow \Gamma(\Sigma, \kappa^{\otimes j} \otimes \bar{\kappa}).$$

REMARK 4.4. Therefore all computations can be performed on the reference Riemann surface (i.e. the 0-fiber).

Let  $\mathcal{L}$  denote the *determinant line bundle* of the family  $\mathcal{T}_j$  over  $B$  [23],

$$(4.18) \quad \mathcal{L} = \lambda(\text{Ker} \mathcal{T}_j)^* \otimes \lambda(\text{Ker} \mathcal{T}_{1-j})^*$$

where we used  $\text{Ker} \mathcal{T}_j^* \simeq (\text{Ker} \mathcal{T}_{1-j})^*$  thanks to the second formula in (4.16).

It has a canonical *holomorphic* structure, see below, even if the family  $\mathcal{T}_j$  is smooth., cf eq.(4.16). If  $\Sigma$  denotes the fixed Riemann surface (0-fiber) then we have the following diagram

$$\begin{array}{ccc} \kappa_\Sigma^{-1} & & \mathcal{L} \\ \downarrow & & \downarrow \\ B \times \Sigma & \xrightarrow{\pi} & B \xrightarrow{\mu} \Sigma \end{array}$$

REMARK 4.5. Since the base  $B$  is contractible the first Chern class of the determinant line bundle  $\mathcal{L}$  is cohomologically trivial that is  $H^2(B, \mathbf{R}) = 0$ .

**4.4. The Quillen metric on  $\mathcal{L}$ .** We shall choose a basis  $\{\gamma_{m,\mu}\}$  (resp.  $\{\beta_{p,\mu}\}$ ),  $m = 1, \dots, N_j$  (resp.  $p = 1, \dots, N_{1-j}$ ) in the kernel of  $\mathcal{T}_{j,\mu}$  (resp.  $\mathcal{T}_{1-j,\mu}$ ) *holomorphic* in  $\mu$ <sup>4</sup> and *independent* of  $\rho$ , with, according to the Riemann-Roch theorem

$$N_j - N_{1-j} = (g - 1)(2j - 1),$$

if  $\Sigma$  is of genus  $g$ .

By construction let  $s$  be the non-vanishing holomorphic section

$$(4.19) \quad s = (\gamma_1 \wedge \dots \wedge \gamma_{N_j})^{-1} \otimes (\beta_1 \wedge \dots \wedge \beta_{N_{1-j}})^{-1},$$

of  $\mathcal{L}$ , see eq.(4.18). Then

DEFINITION 4.6 ([23] or Bost in [1]). The Quillen metric on  $\mathcal{L}$  is defined to be

$$(4.20) \quad \|s\|_Q^2 = (\det'_\zeta \Delta_j) \|s\|_{L_2}^2$$

where the norm

$$\|s\|_{L_2}^2 = \frac{1}{\det \langle \gamma_m | \gamma_n \rangle_{1 \leq m, n \leq N_j}} \frac{1}{\det \langle \beta_p | \beta_q \rangle_{1 \leq p, q \leq N_{1-j}}},$$

---

<sup>4</sup>This can be done locally in  $\mu$ .



all metrics  $\langle \cdot | \cdot \rangle$  (on the 0-fiber) are associated with  $ds^2 = \rho|dz + \mu d\bar{z}|^2$ , and  $\det'_\zeta$  stands for the  $\zeta$ -regularized determinant

$$\ln \det'_\zeta \Delta_j = - \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} dt (\text{Tr } e^{-t\Delta_j} - N_j) .$$

**4.5. The local index theorem for the family  $\mathcal{T}_j$ .** In our case of interest the hypotheses are the following : Let  $\pi : b \times \Sigma \longrightarrow B$  be a smooth proper holomorphic map of Riemann surfaces. Assume that  $\pi$  is locally Kähler, i.e. there is an open covering  $\mathcal{U}$  of  $B$  such that if  $U \in \mathcal{U}$ ,  $\pi^{-1}(U)$  admits a Kähler metric (whose restriction to the fiber  $\Sigma_\mu$ ,  $\mu \in U$  may differ from the Kähler metric (4.7)). Let  $E$  be as above the Hermitian bundle equipped with the Hermitian metric induced by (4.8). Let  $\mathcal{K}$ ,  $\mathcal{E}$  be the curvatures of the holomorphic Hermitian metric on  $\kappa_\Sigma^{-1}$  and  $E$  respectively. Then the theorem states that

**THEOREM 4.7 ([1, 3]).** *Under the above hypotheses, the curvature of the connection associated with the Quillen metric on the determinant line bundle  $\mathcal{L}$  is the component of degree 2 in the following form on  $B$*

$$(4.21) \quad 2i\pi \left[ \pi_* \text{Td}(-\mathcal{K}/2i\pi) \text{Tr}[\exp(-\mathcal{E}/2i\pi)] \right]^{(2)},$$

where  $\pi_* : \Omega_{\text{vc}}^*(B \times \Sigma) \longrightarrow \Omega^{*-2}(B)$  is the “integration along the fiber” in the compact vertical cohomology which commutes with the exterior differentiation  $d_B$  on the base [21].

Consider now for a the quantity

$$(4.22) \quad \Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\rho, 0, 0), \quad \Gamma(\rho, \mu, \bar{\mu}) = \frac{1}{2} \ln \|s\|_Q^2,$$

for a given section (4.19) locally holomorphic in  $\mu$ . Since all bundles are line bundles we can express the equality of differential forms on  $B$  in the following

**COROLLARY 4.8** (e.g. [1]). *The 1-st Chern class of  $\mathcal{L}$  locally represented by*

$$(4.23) \quad c_1(\mathcal{L}) = -\frac{1}{2i\pi} \bar{\partial}_B \partial_B \ln \|s\|_Q^2 = -\frac{1}{2i\pi} \bar{\partial}_B \partial_B (\Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\rho, 0, 0)),$$

*equals*

$$(4.24) \quad c_1(\mathcal{L}) = -[\pi_* \text{Td}(\kappa_\Sigma^{-1}) \text{Ch}(E)]^{(2)} = -\frac{C_j}{12} \pi_* c_1(\kappa_\Sigma^{-1})^2,$$

where the constant  $C_j = 6j(j-1) + 1$ .

For the sake of definiteness, let us consider an analytic submanifold  $B_t \subset B$  parametrized by a finite dimensional holomorphic family  $\mu_t$ , with  $\mu_0 = 0$ , where  $t$  denotes a set of local coordinates of a finite dimensional analytic manifold  $S$ , (e.g. a neighbourhood of the origin in  $\mathbf{C}^n$ ). Following the construction made in [24] let us introduce the map

$$(4.25) \quad \begin{aligned} \Phi : S \times \Sigma &\longrightarrow B \times \Sigma \\ (t, (z, \bar{z})) &\longmapsto (\mu_t, Z_{\mu_t}(z, \bar{z})) \end{aligned}$$

which is holomorphic in  $t$  and a  $C^\infty$ -diffeomorphism of  $(z, \bar{z})$ , so that the coordinate  $Z_{\mu t} \equiv Z_t$  built up from the Beltrami equation (4.6) is chosen to be holomorphic in  $t$ .

Thus, to the complex analytic structure defined by the natural coordinates  $(t, \bar{t}, Z, \bar{Z})$  there corresponds the following splitting of the total differential (4.5)

$$(4.26) \quad D = \mathcal{D} + \bar{\mathcal{D}} = d + d_t + d_{\bar{t}}$$

with

$$(4.27) \quad \mathcal{D} = DZ\partial_Z + Dt\partial_{t|Z}, \quad \mathcal{D}^2 = \mathcal{D}\bar{\mathcal{D}} + \bar{\mathcal{D}}\mathcal{D} = \bar{\mathcal{D}}^2 = 0,$$

where  $\partial_{t|Z}$  denotes partial derivatives with respect to the  $t$  coordinates at constant  $Z, \bar{Z}$ .

Now the first Chern class  $c_1(\kappa_\Sigma^{-1}) \in H^2(B_t \times \Sigma, \mathbf{R})$  of the holomorphic vertical line bundle over  $B_t \times \Sigma$  which corresponds to the Hermitian metric (4.8) is locally represented by the  $D$ -closed form <sup>5</sup> on  $B_t \times \Sigma$  given by

$$(4.28) \quad \begin{aligned} (-2i\pi)c_1(\kappa_\Sigma^{-1}) &= \mathcal{K} = \bar{\mathcal{D}}\mathcal{D} \ln \rho_{Z\bar{Z}} \\ &= (D\bar{Z}\partial_{\bar{Z}} + D\bar{t}\partial_{\bar{t}|Z})(DZ\partial_Z + Dt\partial_{t|Z}) \ln \rho_{Z\bar{Z}} \\ &= (D\bar{Z}DZ\partial_Z\partial_{\bar{Z}} + D\bar{t}DZ\partial_{\bar{t}|Z}\partial_Z + D\bar{Z}Dt\partial_{\bar{Z}}\partial_{t|Z} \\ &\quad + D\bar{t}Dt\partial_{\bar{t}|Z}\partial_{t|Z}) \ln \rho_{Z\bar{Z}}. \end{aligned}$$

Performing the smooth change of local complex coordinates  $(t, \bar{t}, Z, \bar{Z}) \mapsto (t, \bar{t}, z, \bar{z})$  through the expression (4.3) of  $dZ$  here seen as a section over  $B_t$ , we get

$$(4.29) \quad \begin{aligned} (-2i\pi)c_1(\kappa_\Sigma^{-1}) &= [(d\bar{Z} + d\bar{t}\partial_{\bar{t}}\bar{Z})(dZ + dt\partial_t Z)\partial_Z\partial_{\bar{Z}} + d\bar{t}(dZ + dt\partial_t Z)\partial_{\bar{t}|Z}\partial_Z \\ &\quad + (d\bar{Z} + d\bar{t}\partial_{\bar{t}}\bar{Z})dt\partial_{t|Z}\partial_{\bar{Z}} + d\bar{t}dt\partial_{\bar{t}|Z}\partial_{t|Z}] \ln \rho_{Z\bar{Z}} \end{aligned}$$

Using now the chain rule

$$\partial_t = \partial_{t|Z} + (\partial_t Z)\partial_Z$$

where  $\partial_t = \partial_{t|z}$  eq.(4.29) becomes

$$(4.30) \quad (-2i\pi)c_1(\kappa_\Sigma^{-1}) = [d\bar{Z}dZ\partial_Z\partial_{\bar{Z}} + d\bar{Z}dt\partial_t\partial_{\bar{Z}} + d\bar{t}dZ\partial_{\bar{t}}\partial_Z] \ln \rho_{Z\bar{Z}}$$

The “square” of this first Chern class belongs directly to the vertical compact cohomology class  $H_{vc}^4(B_t \times \Sigma, \mathbf{R})$

$$(4.31) \quad c_1(\kappa_\Sigma^{-1})^2 = \frac{1}{2\pi^2} d\bar{Z}dZd\bar{t}dt |\partial_t\partial_{\bar{Z}} \ln \rho_{Z\bar{Z}}|^2.$$

Now the curvature (4.23) gives rise to an obstruction to the holomorphic factorization property, namely

$$\bar{\partial}_B\partial_B (\Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\rho, 0, 0)) \neq 0.$$

Happily, one has

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<sup>5</sup>In the following, the product of differentials is to be understood as the usual wedge product.

THEOREM 4.9 ([6]). *The square of the first Chern class of the vertical holomorphic line bundle over  $B_t \times \Sigma$  is*

$$(4.32) \quad c_1(\kappa_\Sigma^{-1})^2 = \frac{i}{\pi^2} d_{\bar{t}} d_t \chi(\rho, \mu, \bar{\mu}),$$

where  $\chi(\rho, \mu, \bar{\mu})$  is a section in  $\Gamma(B_t, \Omega^{1,1}(\Sigma, \mathbf{R}))$  and depends locally on  $\rho, \mu, \bar{\mu}$ .

PROOF. See [6] formulas (37,38). The expression of  $\chi$  is given in [6] formula (22) (but actually comes from elsewhere [25])

$$(4.33) \quad \begin{aligned} \chi(\rho, \mu, \bar{\mu}) = & \frac{d\bar{z} \wedge dz}{2i} \left( \mu(\mathcal{R} - R) + c.c. - \frac{1}{1 - \mu\bar{\mu}} [(\partial + \Gamma)\mu(\bar{\partial} + \bar{\Gamma})\bar{\mu} \right. \\ & \left. - \frac{1}{2}\bar{\mu}((\partial + \Gamma)\mu)^2 - \frac{1}{2}\mu((\bar{\partial} + \bar{\Gamma})\bar{\mu})^2] \right), \end{aligned}$$

where  $\mathcal{R}$  is the coefficient of a *holomorphic projective connection* [26] and  $\Gamma = \partial \ln \rho$ ,  $\bar{\Gamma} = \bar{\partial} \ln \rho$ ,  $R = \partial \Gamma - \frac{1}{2}\Gamma^2$ .  $\square$

Note that  $\chi(\rho, 0, 0) = 0$ .

COROLLARY 4.10. *One reaches the conclusion that*

$$(4.34) \quad \bar{\partial}_B \partial_B \left( \Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\rho, 0, 0) + \frac{C_j}{12\pi} \pi_* \chi(\rho, \mu, \bar{\mu}) \right) = 0,$$

and there is no longer any dependence on the Weyl factor  $\rho$ ,

$$(4.35) \quad \partial_\rho \left( \Gamma(\rho, \mu, \bar{\mu}) - \Gamma(\rho, 0, 0) + \frac{C_j}{12\pi} \pi_* \chi(\rho, \mu, \bar{\mu}) \right) = 0.$$

Thus holomorphic factorization follows but it is far from being unique. It depends on the existence of the section (4.19) of the determinant line bundle, covariant under diffeomorphisms a problem which has not been addressed.

COROLLARY 4.11. *The first Chern class of the determinant line bundle  $\mathcal{L}$  is then locally expressed as a  $\bar{\partial}_B \partial_B$ -exact form*

$$(4.36) \quad c_1(\mathcal{L}) = \frac{C_j}{12i\pi^2} d_{\bar{t}} d_t \pi_* \chi(\rho, \mu, \bar{\mu}),$$

which can be viewed as another representative of this first Chern class different from that given by the Quillen metric.

This construction can be extended to free fields considered as sections of a holomorphic bundle [7] where besides the Beltrami differential parametrizing the complex structures on the Riemann surface, there is the  $(0,1)$ -component of a connection parametrizing the complex structures of the holomorphic bundle. One reaches the same conclusion as above.

## 5. Concluding remarks

The holomorphic factorization property illustrates the deep relationship between the locality principle of Euclidean field theory and the refined version of the index theorem for elliptic families. The introduction of metrics which cannot be avoided at present, positivity being the main ingredient allowing to sum up the perturbative series, remains compatible with the analyticity.

Accordingly to previous works [6, 7], the lecturer together with M. Knecht and R. Stora resulted in a striking geometrical situation from an application of the local index theorem for families of  $\bar{\partial}$ -operators parametrized by Beltrami differentials. The outcome was that the first Chern class of the determinant line bundle over the space of Beltrami differentials, locally expressed with the aid of Quillen's metric given by

$$(5.1) \quad \bar{\partial}_B \partial_B \ln \|s\|_Q^2 = -\frac{C_1}{6\pi} \bar{\partial}_B \partial_B \int_{\Sigma} \chi,$$

with  $\chi \in \Omega^{1,1}(\Sigma, \mathbf{R})$  a  $(1,1)$ -form on the typical fiber  $\Sigma$  and  $s$  a non-vanishing section of this determinant bundle. Surprisingly this  $(1,1)$ -form depends locally on the Beltrami differentials.

The resulting problem amounts to finding a geometrical reason why the identity (4.32) or (4.36) holds. What is first remarkable is that even the 1-st Chern class  $c_1(\mathcal{L})$  must indeed be  $d_B$ -exact (thanks to the topological triviality) but it is more : it is  $\bar{\partial}_B \partial_B$ -exact of a local quantity ! So one ought to think of a kind of *local holomorphic* triviality. But the computation carried out in [6] shows that the identity (4.36) has a global meaning. According to the definition (4.22) of what can be considered as a resummation of the perturbative series, one has to face a kind of interpolating formula between two Quillen metrics, and the expression for  $\pi_* \chi$  resembles a Bott-Chern transgression formula.

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